Problem of the Month - November 2024 Proposed by: Dr Alok Shukla

The *Taxicab* or *Manhattan distance* between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the plane is defined as:

$$d_{\text{Manhattan}}(P, Q) = |x_1 - x_2| + |y_1 - y_2|.$$

This metric generalizes to higher dimensions as follows. For points $P = (p_1, p_2, \ldots, p_d)$ and $Q = (q_1, q_2, \ldots, q_d)$ in d-dimensional space, the Manhattan distance is:

$$d_{\text{Manhattan}}(P, Q) = \sum_{i=1}^{d} |p_i - q_i|.$$

If n points $P_1, P_2, \ldots, P_n$ are given inside the unit cube $S = [0, 1]^d$, then prove that there always exists a point $P \in S$ such that the sum of the Manhattan distances from P to each of the points $P_1, P_2, \ldots, P_n$ is exactly $\frac{n \cdot d}{2}$. That is, there is a point $P \in S$ that satisfies:

$$\sum_{i=1}^{n} d_{\text{Manhattan}}(P, P_i) = \frac{n \cdot d}{2}.$$